Homework 1
Geometry

Joshua Ruiter

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Note on notation: When I use the symbol $\subset$, it does not imply that the subset is proper. In writing $A \subset X$, I mean only that $a \in A \implies a \in X$, leaving open the possibility that $A = X$. I do not use the symbol $\subseteq$.

Proposition 0.1 (Exercise A.3). *Let X, Y be topological spaces and let $f : X \to Y$ be continuous. Let $(p_i)_{i=1}^\infty$ be a sequence in X such that $p_i \to p$. Then $f(p_i) \to f(p)$ in Y.*

Proof. We need to show that for every neighborhood V of $f(p)$, there exists $N \in \mathbb{N}$ such that $f(p_i) \in V$ for all $i \geq N$. Let V be a neighborhood of $f(p)$. By continuity of f, $f^{-1}(V)$ is an (open) neighborhood of p. Since $p_i \to p$, there exists N such that $p_i \in f^{-1}(V)$ for $i \geq N$. Thus $f(p_i) \in f(f^{-1}(V)) = V$ for all $i \geq N$, so $f(p_i) \to f(p)$. $\square$

Proposition 0.2 (Exercise A.11). *Let X be a Hausdorff space. Then each finite subset of X is closed and each convergent sequence in X has a unique limit.*

Proof. Let $A \subset X$ be a finite subset. We will show that $X \setminus A$ is open. Let $x \in X \setminus A$. By the Hausdorff property, for each $a \in A$, there exist disjoint neighborhoods U_a, V_a such that $a \in U_a, x \in V_a$ with $U_a \cap V_a = \emptyset$. Then let $V = \cap_{a \in A} V_a$. Since A is finite, V is an open neighborhood of x. V must also be disjoint from A, since for each $a \in A$, there is a V_a not including that particular a. Thus V is a neighborhood of x contained in $X \setminus A$, hence $X \setminus A$ is open, so A is closed.

Let $(p_i)_{i=1}^\infty$ be a convergent sequence in X. Suppose as an RAA hypothesis that p, q are distinct limits for p_i, that is, $p_i \to p$ and $p_i \to q$ and $p \neq q$. By the Hausdorff property, there exist neighborhoods P, Q with $p \in P, q \in Q, P \cap Q = \emptyset$. By the convergence of p_i to p and q, there exist N_p and N_q such that $i \geq N_p \implies p_i \in P$ and $i \geq N_q \implies p_i \in Q$. Let $N = \max\{N_p, N_q\}$. Then for $i \geq N$, we have $p_i \in P$ and $p_i \in Q$. But $P \cap Q = \emptyset$, so this is a contradiction. Thus we reject the RAA hypothesis and conclude that p_i cannot have more than one limit. $\square$

Lemma 0.3 (for Exercise A.13). *Let X be a topological space and let $A \subset X$. Then $A = \overline{A}$ if and only if A is closed.*

Proof. Suppose that $A \subset X$ is closed. By definition,

$$\overline{A} = \bigcap_\alpha A_\alpha$$

where A_α is a closed set with $A \subset A_\alpha$. Since A is closed, $A = A_\alpha$ for some α, so $\overline{A} \subset A$. Since A is contained in each A_α, $A \subset \overline{A}$. Hence $A = \overline{A}$.

Conversely, suppose that $A = \overline{A}$. Since $\overline{A}$ is an intersection of closed sets, its complement is a union of open sets, so its complement is closed, so $\overline{A}$ is closed. Hence A is closed. $\square$

Lemma 0.4 (for Exercise A.13). *Let X be a topological space and let $A \subset X$. Then $A = \operatorname{Int} A$ if and only if A is open.*

Proof. Suppose that $A \subset X$ is open. Then A is an open set contained within A, so $A \subset \operatorname{Int} A$. Since $\operatorname{Int} A$ is a union of sets contained within A, we have $\operatorname{Int} A \subset A$. Thus $A = \operatorname{Int} A$.

Conversely, suppose that $A = \operatorname{Int} A$. Since $\operatorname{Int} A$ is a union of open sets, it is open, thus A is open. $\square$

Proposition 0.5 (Exercise A.13). *Let X be a first-countable space, let $A \subset X$, and let $x \in X$.*

1. *$x \in \overline{A}$ if and only if x is a limit of a sequence in A.*
2. *$x \in \operatorname{Int} A$ if and only if every sequence in X converging to x is eventually in A.*
3. *A is closed in X if and only if A contains every limit of every convergent sequence in A.*
4. *A is open in X if and only if every sequence in X converging to a point in A is eventually in A.*

Proof. First we prove (1). Let $x \in \overline{A}$. If $x \in A$, then $(x, x, \ldots)$ is a sequence in A converging to x. If $x \notin A$, then $x \in \overline{A} \setminus A$. Since X is first-countable, there is a countable collection $\mathcal{B}_x$ of neighborhoods of x such that every neighborhood of x contains at least one $B \in \mathcal{B}_x$. We claim that each neighborhood $B_n \in \mathcal{B}_x$ intersects A. If not, then $(B_n)^c$ is a closed set containing A with $x \notin (B_n)^c$, but this is impossible since $x \in \overline{A}$. We order the countable neighborhoods of x as $\{B_n\}_{n=1}^\infty$. Then for each $k \in \mathbb{N}$, we choose

$$x_k \in \left(\bigcap_{n=1}^{k} B_n \right) \cap A$$

We claim that $x_k \to x$. Let U be a neighborhood of x. Since $\mathcal{B}_x$ is a neighborhood basis of x, there exists N such that $x \in B_N \subset U$. Then $x_n \in B_N$ for all $n \geq N$, so $x_k \to x$.

Now we show the other direction of (1). Suppose that $x \in X$, and there exists a sequence $(x_n)_{n=1}^\infty$ in A with $x_n \to x$. Let C be any closed set with $A \subset C$, and suppose that $x \notin C$. Then $x \in C^c$, so C^c is an open neighborhood of x. Since $x_n \to x$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $x_n \in C^c$. However, $x_n \in A \subset C$ for all n, so this is a contradiction. Hence $x \in C$, for all closed sets C containing A. Thus x is in the intersection over all such closed sets, which is precisely $\overline{A}$.

Now we prove (2). First suppose that $x \in \operatorname{Int} A$. Then there exists an open set U with $x \in U \subset A$. Let x_n be a sequence in X that converges to x. Then there exists $N \in \mathbb{N}$ such that $n \geq \mathbb{N}$ implies $x_n \in U \subset A$, so x_n is eventually in A.

Now we prove the other direction of (2). Suppose that every sequence in X converging to x is eventually in A. Let $\mathcal{B}_x = \{B_n\}_{n=1}^{\infty}$ be an ordered, countable neighborhood basis for x. For $k \in \mathbb{N}$, choose an $x_k \in \bigcap_{n=1}^{k} B_n$, so $x_k \to x$. Then by hypothesis, x_k is eventually in A. We claim that for some k, $\bigcap_{n=1}^{k} B_n \subset A$. If not, then we can construct a sequence by choosing $y_k \in \left(\bigcap_{n=1}^{k} B_n\right) \setminus A$. For this sequence (y_k), we have $y_k \to x$, but $y_k \notin A$ for all k, which contradicts the hypothesis (that all sequences converging to x are eventually in A). Thus $\bigcap_{n=1}^{k} B_n \subset A$ for some k, and this intersection in an open neighborhood of x contained within A. Hence $x \in \operatorname{Int} A$.

Now we prove (3). Suppose that $A \subset X$ is closed. By the above lemma, $A = \overline{A}$. Then by (1), $x \in A$ if and only if x is a limit of a sequence of points in A, so A contains every limit of every convergent sequence of points in A.

Conversely, if A contains every limit of every convergent sequence in A, then $\overline{A} \subset A$. Then if $\overline{A} = \bigcap_\alpha A_\alpha$, we have $\bigcap_\alpha A_\alpha \subset A$. Since $A \subset A_\alpha$ for each α, then $A \subset \bigcap_\alpha A_\alpha$, so we have two-way containment, and hence $A = \overline{A}$. Then by the above lemma, A is closed.

Now we prove (4). Suppose that A is open. Then $A = \operatorname{Int} A$ by the above lemma. Let $x \in A$. Then by (2), every sequence in X converging to $x \in A$ is eventually in A.

Conversely, suppose that for $x \in A$, every sequence in X converging to x is eventually in A. Then by (2), $x \in \operatorname{Int} A$. Thus $A \subset \operatorname{Int} A$, so $A = \operatorname{Int} A$ since we know that $\operatorname{Int} A \subset A$. Hence A is open. □

Proposition 0.6 (Exercise A.15)**.** *The set of all open balls in $\mathbb{R}^n$ with rational radius and centers with rational coordinates is a countable basis for $\mathbb{R}^n$ with the Euclidean metric topology. Thus $\mathbb{R}^n$ is second-countable.*

Proof. We need to show that every open subset of $\mathbb{R}^n$ is a union of a collection of such rational balls. (Clearly this basis is countable.) Let $U \subset \mathbb{R}^n$ be open. Then there exists $r > 0$ such that $B(x, r) \subset U$. There exists some $q \in \mathbb{Q}$ with $0 < q < r/2$, so $B(x, q) \subset B(x, r) \subset U$.

Along each dimension from $i = 1$ to $i = n$, there is a line segment of the form $(x^i - q/\sqrt{n}, x^i + q/\sqrt{n})$ contained in $B(x, q)$ (note that the superscript i is an index, not an exponent). From each interval $(x^i - q/\sqrt{n}, x^i + q/\sqrt{n})$ we choose a $p^i \in \mathbb{Q}$, and construct $p = (p^1, p^2, \dots p^n) \in \mathbb{Q}^n \subset \mathbb{R}^n$. We claim that $x \in B(p, q)$. Notice that

$$||x - p|| = \sqrt{\sum_{i=1}^{n} (x^i - p^i)^2} < \sqrt{\sum_{i=1}^{n} (q/\sqrt{n})^2} = \sqrt{\sum_{i=1}^{n} q^2/n} = \sqrt{q^2} = q$$

So we have $||x - p|| < q$, which is equivalent to $x \in B(p, q)$. We also claim that $B(p, q) \subset B(x, r)$. Since $q < r/2$, the diameter of $B(p, q)$ is less than r, and since $B(p, q)$ contains x, it cannot contain any point of a distance more than r away from x. Thus $B(p, q) \subset B(x, r) \subset U$.

Thus for every open set $U \subset \mathbb{R}^n$, and every $x \in U$, there is a ball $B(p_x, q_x)$ with rational radius q_x and rational center coordinates $(p^1, \dots p^n)$ with $x \in B(p_x, q_x) \subset U$. We can then write U as

$$U = \bigcup_{x \in U} B(p_x, q_x)$$

Thus every open subset of $\mathbb{R}^n$ can be written as a union of balls with rational radius and rational coordinates, so these balls form a (countable) basis for $\mathbb{R}^n$. Hence $\mathbb{R}^n$ is second-countable. $\square$

Proposition 0.7 (Exercise A.42b). *Every path-connected space is connected.*

Proof. Let X be path-connected. Suppose that X is not connected. Then there exist disjoint open sets U, V whose union is X. Choose $x \in U, y \in V$. Since X is path-connected, there is a path $\gamma : [0,1] \to X$ with $\gamma(0) = x, \gamma(1) = y$. By continuity of γ, $\gamma^{-1}(U)$ and $\gamma^{-1}(V)$ are open in $[0,1]$. We also know that $\gamma^{-1}(U) \cup \gamma^{-1}(V) = [0,1]$ and $\gamma^{-1}(U) \cap \gamma^{-1}(V) = \emptyset$. Since $0 \in \gamma^{-1}(U)$ and $1 \in \gamma^{-1}(V)$, neither is empty. But this implies that $[0,1]$ is disconnected, which is false. Hence X must be connected.

$\square$